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CREEP FLOW AND PRESSURE RELAXATION IN BUBBLY MEDIUM

A. N. SALAMATIN
Kazan State University, 18 Lenin St, Kazan 420008, Russia

and

P. DUVAL
Laboratoire de Glaciologie et Géophysique de l'Environnement, Rue Molière, B.P.96, 38402 St-Martin-d'Hères Cedex, France

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Abstract—General equations governing the slow creep motion of a nonlinear viscous, incompressible medium containing a large number of small gas bubbles are analyzed on the basis of asymptotic averaging methods for periodic structures. Special attention is paid to account for the interaction of bubble compression (decompression) relaxation and deviatoric macro deformations in the two-phase system. The corresponding approximate rheological relations and averaged macro-scale mass and momentum balance equations are derived. The relationship between gas-medium pressure drop and volume expansion (compression) rate, as well as the one between deviatoric macro-stresses and macro-strain rates are numerically examined in application to bubbly ice rheology. Copyright © 1996 Elsevier Science Ltd

NOTATION

a	Radius of bubbles
$\mathbf{B}_i$	Vector of periodicity
c	Gas volume concentration
$\dot{\mathbf{e}}$	Strain rate tensor in fluid matrix
$\dot{\mathbf{E}}$	Deviator of macro-scale strain rates
F	Function defined by eqns (30), (35)
$\mathbf{g}$	Gravity vector
G_α, H	Functions defined by eqns (41), (42); $\alpha = x, y, z$
$K_0(\partial K_0)$	Cell of periodicity (boundary of the cell)
K_m	Fluid matrix within cell
l	Length of edge of equivalent cubic bubble
L	Macro scale of bubbly medium flow
$\mathbf{n}$	Unit normal at gas–fluid interface
p	Pressure
P	Excess pressure in cell
q_0	Macroscopic volumetric expansion rate
q_α	Principal components of $\dot{\mathbf{E}}$
Q	Activation energy of ice
$\mathbf{r}$	Radius-vector on macro scale (in $\mathbf{r}$-space)
$\mathbf{R}$	"Fast" radius-vector on micro scale (in $\mathbf{R}$-space)
R	Radial coordinate
R_b	Radius of bubble in $\mathbf{R}$-space
R_s	Gas constant
R_0	Radius of sphere of unit volume in $\mathbf{R}$-space
$\dot{\mathbf{S}}$	Micro-scale fluctuations of strain rate tensor
S	Distance from cell center along α-sector axis
t	Time
$\mathbf{T}$	Macro stress tensor
T	Temperature (K)
$\mathbf{v}$	Velocity vector
v_b	Volume of gas bubble in $\mathbf{r}$-space
V_b	Volume of gas bubble in $\mathbf{R}$-space
V_0	Volume of cell K_0
$\mathbf{W}$	Velocity fluctuations within cell
X, Y, Z	Fast Cartesian coordinates
α	Creep index, exponent

$\gamma(\Gamma)$	Interface surface in $\mathbf{r}$-($\mathbf{R}$-)space
δ	Identity tensor
$\tilde{\Delta}$	Distance between bubbles
ε	Small parameter, ratio: Δ/L
η	Rheological coefficient in eqn (1)
θ	Coordinate angle
κ	Tuning parameter, form factor in spherical and cubical approximations
μ_1, μ_2	Rheological coefficients of ice in eqn (45)
ν	Normalized distance from cell center
ρ	Density
$\boldsymbol{\sigma}$	Stress tensor
$\boldsymbol{\tau}$	Deviator of $\boldsymbol{\sigma}$
φ	Coordinate angle
II, III	Second and third invariants of tensor
$\langle\cdot\rangle$	Averaging operation
∇	Nabla operator

Superscripts

s	Symmetrization operation

Subscripts

a	Apparent value
b	Bubble
m	Fluid matrix, medium
$r(R)$	Differentiation with respect to $\mathbf{r}(\mathbf{R})$, component
s	Standard conditions
x, y, z	Components in Cartesian coordinate system
$\alpha, \beta, \gamma, \theta, \varphi$	Directional averaging, components
$0, 1, 2$	Coefficients in asymptotic expansions

1. INTRODUCTION

Gas–liquid mixtures and other fluid bubbly mediums are widely spread in the nature and used in industrial technologies. Even if the continuous phase is incompressible, the whole two-phase system, being involved in a flow process, can change its volume (bulk density) under compression or decompression due to the presence of gas bubbles.

Mathematical description of rheological behavior of such mediums is the general problem in multiphase fluid mechanics (Nigmatulin, 1987). It includes appropriate formulation of the Rayleigh–Lamb equation (Plesset and Prosperetti, 1977; Nigmatulin, 1987), which relates the phase pressure difference to the volume strain rate, inertia effects in the fluid and so on.

A special case of the problem arises in modelling slow creep flows of nonlinear viscous, bubbly mediums. In these processes the drop in pressure between the gas and the fluid matrix appears to be the main driving force of the volume expansion (compression), and a principal peculiarity is the interaction of volumetric and deviatoric deformations because of the viscous nonlinearity of the continuous, fluid phase. The authors have encountered the problem in their study of dry bubbly ice densification processes in ice sheets (Salamatin *et al.*, 1985; Pimienta *et al.*, 1987; Salamatin, 1991). The earlier investigations were undertaken by Bader (1965), Gow (1968), and Brown (1979). Similar phenomena can be observed, for instance, during the closure of a bore-hole drilled in a glacier, in volcanic lava flows, or in pressure sintering of powder compacts. The latter case was examined by Wilkinson and Ashby (1975). A brief review of theoretical results and mathematical approaches used for predicting pressure (density) relaxation and expansion rates in these situations has recently been published by Salamatin and Lipenkov (1993). All of the models deal with volumetric deformations of a bubbly medium under the universal compression (decompression) without taking into account the deviatoric part of the strains, which can be significant, if not dominant. Moreover, the above-mentioned processes of bubbly ice densification and pressure sintering in ceramics production are typical examples of the uniaxial compression, which results in non-zero deviatoric strain rates of the same order as the densification rate. Hence, the study of the interaction between the main flow and the density (pressure) relaxation in the nonlinear viscous, bubbly medium may be of primary interest in applications.

The size of bubbles in the two-phase system is hereafter assumed to be small in comparison with the macro scale of its motion. This permits cmployment averaging (homogenization) methods for description of bubbly medium behavior. General problems of averaging in mechanics of multiphase systems were discussed, for instance, by Nigmatulin (1987). Different mathematical approaches to construction of macrocontinual models were analyzed by Salamatin (1987). In all these theoretical methods, a certain pattern of the "inner", micro-scale structure of the suspension (i.e., the scheme of bubble locations in the continuous phase) is accepted *a priori* to simulate the real situation. Here we shall assume, after Salamatin *et al.* (1985), that the bubbly medium is periodic. Many natural and artificial gas–liquid mixtures are regular to some extent, and the latter assumption is a good approximation to study their rheological behavior on the macro-scale level. At the same time the asymptotic averaging analysis yields in this case the most consistent and rigorous procedure. The theory of homogenization for periodic structures based on perturbation methods has been developed and expounded, in particular, by Bensoussan *et al.* (1978), Sanchez-Palencia (1980) and Bahvalov and Panasenko (1984).

All further considerations in this paper use this instrument and are aimed to deduce averaged balance equations and rheological relations governing slow creep flow of nonlinear viscous, bubbly medium.

2. GENERAL EQUATIONS AND ASYMPTOTIC APPROACH

Let us consider a two-phase system consisting of a large number of discrete bubbles in continuous fluid medium (matrix), which is isotropic, incompressible and follows the quasi-linear non-Newtonian flow law:

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta}+\boldsymbol{\tau}, \quad \boldsymbol{\tau} = 2\eta(4II_e)\dot{\mathbf{e}}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, p, the pressure, $\boldsymbol{\tau}$, the stress deviator, $\boldsymbol{\delta}$, the identity tensor and η the rheological coefficient (apparent viscosity) depending on the second invariant II_e of the strain rate tensor $\dot{\mathbf{e}}$ (deviator) and temperature T.

By definition,

$$\dot{\mathbf{e}} = (\nabla\mathbf{v})^s, \quad II_e = 0.5\dot{\mathbf{e}}:\dot{\mathbf{e}}. \tag{2}$$

Here $\mathbf{v}$ is the velocity vector, ∇ is the differential nabla operator, superscript "s" denotes symmetrization.

The creep noninertial flow of the fluid phase is governed by Stokes equations:

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{g}\rho_m = 0, \quad \nabla\cdot\mathbf{v} = 0, \tag{3}$$

where $\mathbf{g}$ is the gravity vector, ρ_m is the density of the medium.

Mass forces and the constitutive part of the stretching stress in the gas are negligible. Hence, for the bubble pressure p_b we have

$$\nabla p_b = 0, \tag{4}$$

and at the gas–liquid interface γ_b we write

$$\boldsymbol{\sigma}\cdot\mathbf{n}_{|\gamma_b} = -\mathbf{n}p_b, \tag{5}$$

where $\mathbf{n}$ is the unit normal directed outside the fluid matrix.

Hereafter we additionally assume that there is no phase change within the system and the interface is impermeable for gas. If the gas is ideal, then for each single bubble of volume v_b the mass balance equation and the mass transfer condition at the interface take the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_b v_b/T) = 0, \quad \frac{\mathrm{d}v_b}{\mathrm{d}t} = -\int_{\gamma_b} \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\gamma, \tag{6}$$

where t is the time.

Equations (1)–(6) are conventional: see, for instance, Happel and Brenner (1965), Astarita and Marrucci (1974). These equations also formed the theoretical basis for the earlier studies by Salamatin and Lipenkov (1993). In general, the stated problem is very complicated and difficult. It might be efficiently solved only in some specific cases with all simplifying peculiarities of the concrete process thoroughly taken into consideration. Here we are going to exploit the fact that the size a (radius) of bubbles and the characteristic distance Δ between neighbor gas inclusions are considerably less than the macro scale L of the whole deforming domain. Thus, the ratio $\varepsilon = \Delta/L$ is a small parameter in the above problem and the flow of the bubbly medium can be analyzed asymptotically as $\varepsilon \to 0$. Further, regarding the length scale L as a typical one for the macrocontinual description, we take $L \sim 1$. Obviously, $a \sim \varepsilon c^{1/3}$, where c is the volume concentration of the gas (porosity): $c \sim a^3/\Delta^3$.

Following the principal line of perturbation methods, let us distinguish after Bensoussan *et al.* (1978) and Sanchez-Palencia (1980) in the given reference frame the radius-vector $\mathbf{r}$, which is typical for the macro scale L, and $\mathbf{R} = \mathbf{r}/\varepsilon$, which is the "fast" (or "near-field") radius-vector related to the micro scale Δ. Correspondingly, all hydrodynamic characteristics take the form of the double-scaled expansions:

$$\begin{aligned} p &= p_0(\mathbf{r},\mathbf{R}) + \varepsilon p_1(\mathbf{r},\mathbf{R}) + O(\varepsilon^2), \\ \mathbf{v} &= \mathbf{v}_0(\mathbf{r}) + \varepsilon\mathbf{v}_1(\mathbf{r},\mathbf{R}) + \varepsilon^2\mathbf{v}_2(\mathbf{r},\mathbf{R}) + O(\varepsilon^2), \ldots \quad \mathbf{R} = \mathbf{r}/\varepsilon, \varepsilon \to 0. \end{aligned} \tag{7}$$

The nabla operator in eqns (2)–(4) can then be represented as the sum

$$\nabla = \nabla_r + \varepsilon^{-1}\nabla_R, \tag{8}$$

where ∇_r and ∇_R operate separately with respect to the variables $\mathbf{r}$ and $\mathbf{R}$.

Substituting eqns (7) and (8) into eqns (1) and (2), we have

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\mathbf{e}}_0 + \varepsilon\dot{\mathbf{e}}_1 + O(\varepsilon^2) \equiv (\nabla_r\mathbf{v}_0 + \nabla_R\mathbf{v}_1)^s + \varepsilon(\nabla_r\mathbf{v}_1 + \nabla_R\mathbf{v}_2)^s + O(\varepsilon^2), \\ II_e &= II_{e0} + \varepsilon\dot{\mathbf{e}}_0 : \dot{\mathbf{e}}_1 + O(\varepsilon^2), \\ \eta(4II_e) &= \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2) \equiv \eta(4II_{e0}) + 4\varepsilon\eta'(4II_{e0})\dot{\mathbf{e}}_0 : \dot{\mathbf{e}}_1 + O(\varepsilon^2), \\ \boldsymbol{\sigma} &= -p_0\delta + 2\eta_0\dot{\mathbf{e}}_0 + \varepsilon[-p_1\delta + 2(\eta_0\dot{\mathbf{e}}_1 + \eta_1\dot{\mathbf{e}}_0)] + O(\varepsilon^2). \end{aligned} \tag{9}$$

These relations yield the following asymptotic form of eqns (3) and (4):

$$\begin{aligned} \varepsilon^{-1}[-\nabla_R p_0 + 2\nabla_R\cdot(\eta_0\dot{\mathbf{e}}_0)] - \nabla_r p_0 + 2\nabla_r\cdot(\eta_0\dot{\mathbf{e}}_0) - \nabla_R p_1 + 2\nabla_R\cdot(\eta_0\dot{\mathbf{e}}_1 + \eta_1\dot{\mathbf{e}}_0) + \mathbf{g}\rho_m &= O(\varepsilon), \\ \nabla_r\cdot\mathbf{v}_0 + \nabla_R\cdot\mathbf{v}_1 &= O(\varepsilon), \\ \varepsilon^{-1}\nabla_R p_{b0} + \nabla_R\nabla_r p_{b0} + \nabla_R p_{b1} &= O(\varepsilon). \end{aligned} \tag{10}$$

Analogous expansions are also evident for eqns (5) and (6), and we come to the sequence of linked problems to determine the coefficients in the asymptotic series (7).

It should be noted here that the above assumption of $\mathbf{v}_0$ not being dependent on $\mathbf{R}$ is introduced in eqns (7) just to simplify the considerations without any loss of generality. Otherwise, the principal term $\varepsilon^{-1}(\nabla_R\mathbf{v}_0)^s$ would appear in the expansion for the tensor $\dot{\mathbf{e}}$, and the stress tensor $\boldsymbol{\sigma}$ would become unbounded as $\varepsilon \to 0$ (see eqns (9)).

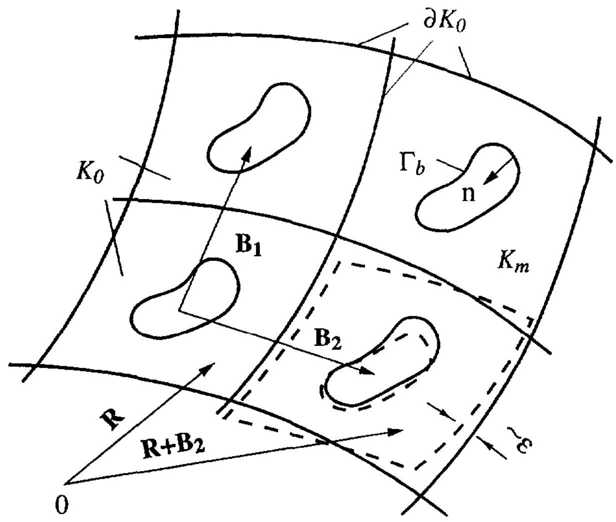

Fig. 1. A sketch of a locally periodic structure.

3. CELL PROBLEM AND AVERAGED BALANCE EQUATIONS FOR LOCALLY PERIODIC STRUCTURES

The next step is to examine the principal perturbation part of the model (1)–(6). From eqns (9) and (10), when the fast variable **R** is within the fluid matrix, one obtains

$$-\nabla_R p_0 + 2\nabla_R \cdot (\eta_0 \dot{\mathbf{e}}_0) = 0, \quad \nabla_r \cdot \mathbf{v}_0 + \nabla_R \cdot \mathbf{v}_1 = 0, \tag{11}$$

and the equation $\nabla_R p_{b0} = 0$ within the gas phase means that p_{b0} depends only on **r**.

At the gas–liquid interface Γ_b in **R**-space eqn (5) with the secondary terms of ε-order omitted takes the form

$$(-p_0\mathbf{n} + 2\eta_0\dot{\mathbf{e}}_0 \cdot \mathbf{n})_{|\Gamma_b} = -\mathbf{n}p_{b0}. \tag{12}$$

For given values $\nabla_r \cdot \mathbf{v}_0$ and p_{b0} eqns (11), (12) determine p_0 and $\mathbf{v}_1$ as functions of **R**. The radius-vector **r** in the problem should be considered as a parameter.

The system of eqns (11), (12) describes the quasi-hydrostatic interaction of gas inclusions through the fluid phase and in case of arbitrary disposition of bubbles is not much easier to solve, if compared with the original basic model (1)–(6). The main advantage can be derived from the above asymptotic analysis, when the bubbly medium has a regular structure.

Actually, let us assume that the bubbles are similar in shape and are located in the fluid almost periodically (see Fig. 1), i.e., for each bubble we can determine in **R**-space three non-coplanar vectors $\mathbf{B}_i$ ($i = 1, 2, 3$) such that the $\mathbf{B}_i$-shift of the space ($\mathbf{R} \to \mathbf{R} + \mathbf{B}_i$) within a finite vicinity of the bubble is almost identical and leaves the structure of the bubbly medium unchanged with the accuracy to ε-order deformations. This assumption means that p_0 and $\mathbf{v}_1$ in eqns (11), (12) and any other characteristic ϕ defined on the basis of the expansions (7), being continuous, are approximately periodic functions in the **R**-space:

$$\phi(\mathbf{r}, \mathbf{R}) = \phi(\mathbf{r}, \mathbf{R} \pm \mathbf{B}_i) + O(\varepsilon), \quad i = 1, 2, 3. \tag{13}$$

Eventually each bubble is confined into a cell K_0 with edges equal in length and parallel to $\mathbf{B}_i$, so that $p_0(\mathbf{r}, \mathbf{R})$ and $\mathbf{v}_1(\mathbf{r}, \mathbf{R})$ are approximately the K_0-periodic solution of eqns (11), (12). This solution is unique, and $\mathbf{v}_0(\mathbf{r})$ has the meaning of the averaged velocity of the medium (see: Sanchez-Palencia, 1980; Bahvalov and Panasenko, 1984), if the extra condition holds:

$$\langle \mathbf{v}_1 \rangle_m = 0, \tag{14}$$

where $\langle \cdot \rangle_m$ is the integral over the domain K_m of the cell K_0 occupied by the fluid matrix. The boundary value problem (11)–(14) is called the cell problem.

The averaging procedure for periodic structures has already been referred to be thoroughly elaborated. Accordingly, to determine the averaged characteristics for the bubbly medium flow (the principal terms in series (7)) and the corresponding averaged balance equations one has to examine the next, second-order terms in the expansions (10) and in the boundary condition (5).

Thus, we write

$$\begin{gathered} -\nabla_R p_1 + 2\nabla_R \cdot (\eta_0 \dot{\mathbf{e}}_1 + \eta_1 \dot{\mathbf{e}}_0) - \nabla_r p_0 + 2\nabla_r \cdot (\eta_0 \dot{\mathbf{e}}_0) + \mathbf{g}\rho_m = 0, \\ \nabla_R p_{b1} + \nabla_r p_{b0} = 0, \\ (-p_1 \mathbf{n} + 2\eta_0 \dot{\mathbf{e}}_1 \cdot \mathbf{n} + 2\eta_1 \dot{\mathbf{e}}_0 \cdot \mathbf{n})_{|\Gamma_b} = -n p_{b1|\Gamma_b}. \end{gathered} \tag{15}$$

The main point that should be stated here is that eqns (15) are formulated for approximately periodic functions p_0, p_1 and $\mathbf{v}_1$, $\mathbf{v}_2$. The latter ones enter eqns (15) via $\dot{\mathbf{e}}_0$ and $\dot{\mathbf{e}}_1$. Due to this fact, after integrating the first eqn (15) over the fluid phase K_m and applying Gauss theorem to transform its first two terms on the left-hand side into surface integrals, we can omit the integral along the cell boundary ∂K_0, since it is of the order of $O(\varepsilon)$. Further, let us integrate the second eqn (15) over the bubble domain. Finally, adding the two equations and using the boundary condition (15), we come to the following result:

$$-\nabla_r(\langle p_{b0} \rangle_b + \langle p_0 \rangle_m) + 2\nabla_r \cdot \langle \eta_0 \dot{\mathbf{e}}_0 \rangle_m + \langle \mathbf{g}\rho_m \rangle_m = O(\varepsilon), \tag{16}$$

which is the averaged momentum balance equation for the bubbly medium; $\langle \cdot \rangle_b$ denotes integration over a bubble.

In its turn, the second eqn (6) after substitution of eqns (7) gives a relation determining the volume of the bubble $V_b = v_b/\varepsilon^3$ on the micro scale in the cell K_0:

$$\frac{\mathrm{d}V_b}{\mathrm{d}t} = V_b \nabla_r \cdot \mathbf{v}_0 + \int_{\Gamma_b} \mathbf{v}_1 \cdot \mathbf{n}\, \mathrm{d}\Gamma + O(\varepsilon). \tag{17}$$

The right-hand side in the latter equality has been transformed in accordance with Gauss theorem.

Again, taking into account the approximate K_0-periodicity of $\mathbf{v}_1$ (see eqn (13)) and using Gauss theorem to integrate the fluid-phase continuity eqn (10), one obtains:

$$\int_{\Gamma_b} \mathbf{v}_1 \cdot \mathbf{n}\, \mathrm{d}\Gamma = \langle \nabla_r \cdot \mathbf{v}_0 \rangle_m + O(\varepsilon).$$

Hence, eqn (17) takes the form:

$$\mathrm{d}V_b/\mathrm{d}t = V_0 \nabla_r \cdot \mathbf{v}_0 + O(\varepsilon), \tag{18}$$

where V_0 is the volume of the cell K_0. Equation (18) has the meaning of the averaged, fluid-mass balance equation.

Now, to conclude with the averaged (homogeneous) description of the bubble medium flow let us introduce new macro-scale variables:

$$c = V_b/V_0 \text{ — gas volume concentration;}$$

$$p_m = (\langle p_{b0} \rangle_b + \langle p_0 \rangle_m)/V_0 \text{ — averaged pressure;}$$

$$\mathbf{T} = 2\langle \eta_0 \dot{\mathbf{e}}_0 \rangle_m / V_0 \text{ — macroscopic stress tensor.} \tag{19}$$

By definition, for incompressible medium we have:

$$\mathrm{d}V_b/\mathrm{d}t = \mathrm{d}V_0/\mathrm{d}t; \quad \mathrm{d}[V_0(1-c)]/\mathrm{d}t = 0.$$

Correspondingly, the first eqn (6) and eqns (16), (18) can be rewritten without negligible terms of the order of $O(\varepsilon)$ as a system of macrocontinual mass and momentum balance equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_b c}{T(1-c)}\right) = 0; \quad \frac{\mathrm{d}c}{\mathrm{d}t} = (1-c)\,\nabla\cdot\mathbf{v}_0,$$

$$\nabla p_m = \nabla\cdot\mathbf{T} + (1-c)\rho_m\mathbf{g}. \tag{20}$$

The subscript "0" in the denotation p_{b0} and the subscript "r" in the ∇_r-operator are omitted, since all characteristics in eqns (20) depend only on the macro-scale space variable $\mathbf{r}$.

It is necessary to emphasize that the model (20) is not complete and we need to solve the cell problem (11)–(14) to formulate explicitly the necessary constitutive eqns (19) relating $p_m - p_b$ and $\mathbf{T}$ to $\nabla\cdot\mathbf{v}_0$ and $\nabla\mathbf{v}_0$. This result common for the theory of periodic structures is shown here to be valid at weaker constraints of local (approximate) periodicity of bubbly medium.

4. CELL PROBLEM ANALYSIS AND APPROXIMATE SOLUTIONS

Now let us designate as P the excess pressure in the fluid, continuous phase:

$$P = p_0 - p_m$$

and introduce additional notations for the macroscopic volumetric expansion rate:

$$q_0 = \nabla\cdot\mathbf{v}_0/3$$

and for the deviator of the macro-scale strain rate tensor:

$$\dot{\mathbf{E}} = (\nabla\mathbf{v}_0)^s - q_0\boldsymbol{\delta}.$$

Let us also place the origin of the reference frame of the $\mathbf{R}$-space into the geometric center of the cell K_0 under consideration. Then $\dot{\mathbf{e}}_0$ can be presented as the sum:

$$\dot{\mathbf{e}}_0 = \dot{\mathbf{E}} + \dot{\mathbf{S}}, \tag{21}$$

and its second invariant

$$II_{e0} = II_E + II_S + \dot{\mathbf{E}} : \dot{\mathbf{S}},$$

where

$$\dot{\mathbf{S}} = (\nabla\mathbf{W})^s, \; \mathbf{W} = \mathbf{v}_1 + q_0\mathbf{R}.$$

The normalized micro-scale fluctuations of the fluid velocity $\mathbf{W}$ and the excess pressure P within K_m are the solution of the modified cell problem (11)–(14):

$$-\nabla P + 2\nabla \cdot (\eta_0 \dot{\mathbf{e}}_0) = 0, \quad \nabla \cdot \mathbf{W} = 0, \quad \mathbf{R} \in K_m;$$
$$(-P\mathbf{n} + 2\eta_0 \dot{\mathbf{e}}_0 \cdot \mathbf{n})_{|\Gamma_b} = -(p_b - p_m)\mathbf{n}_{|\Gamma_b};$$
$$\langle \mathbf{W} \rangle_m = 0. \tag{22}$$

On the opposite faces of the cell boundary ∂K_0, as a consequence of the K_0-periodicity of the functions p_0 and $\mathbf{v}_1$, we have

$$P(\mathbf{R}) = P(\mathbf{R} \pm \mathbf{B}_i), \quad \mathbf{W}(\mathbf{R}) \pm q_0 \mathbf{B}_i = \mathbf{W}(\mathbf{R} \pm \mathbf{B}_i), \quad i = 1, 2, 3. \tag{23}$$

The subscript "R" at ∇_R-operator in eqns (21)–(23) is omitted and hereafter in this section all differential operations should be understood as those in the **R**-space.

By definition, since q_0 and $\dot{\mathbf{E}}$ are considered constant (i.e., independent of **R**), from eqns (19) it is obvious that

$$p_m - p_b = \langle P \rangle_m / V_b,$$
$$\mathbf{T} = 2(\langle \eta_0 \rangle_m \dot{\mathbf{E}} + \langle \eta_0 \dot{\mathbf{S}} \rangle_m)/V_0. \tag{24}$$

The primary goal of analyzing the boundary value problem (21)–(23) is to express the variables P and $\mathbf{W}$ (or $\dot{\mathbf{S}}$) as functions of **R**, q_0 and $\dot{\mathbf{E}}$ in order to substitute them into eqns (24) and, thus, to obtain the constitutive macro-scale relations for $p_m - p_b$ and **T** in the model (20).

With this in mind, it is helpful to note that the right-hand sides in eqns (24), which depend on the solution of the cell problem, are presented in an averaged, integral form. This permits us to expect them to be correct and sufficiently accurate even for plausible approximations for P and $\dot{\mathbf{S}}$. This peculiarity is another reason to use simplified patterns of bubbles location, such as periodic structures, and perturbation methods to describe rheological behavior of real multiphase systems on the macro-scale level, on the average, but here arises a circumstance that should be discussed more thoroughly.

Actually, the averaging procedure in eqns (24) depends on the orientation of the cell K_0 (on the vector triad $\mathbf{B}_i$, $i = 1, 2, 3$) relative to the principal axes of the tensor $\dot{\mathbf{E}}$. Thus, in general, for non-symmetric cells and inclusions we come to an anisotropic macro-continual model for the two-phase mixture. So, if the system we study shows isotropic behavior on the macro-scale level, its imaginable micro-scale structure ought to be specially "designed" in the framework of the local periodicity approximation. Thus, confining further investigations to macroisotropic two-phase mediums, let us assume that:

(1) gas bubbles are spheric and are located in geometric centers of the cells;

(2) each cell has approximately a cubic form identical to that of its neighbors with the accuracy to ε-order relative deformations;

(3) edges of each cubic cell (vectors $\mathbf{B}_i$, $i = 1, 2, 3$) are directed along the principal axes of the macro-strain rate tensor $\dot{\mathbf{E}}$.

This provides both the local periodicity of the structure and its apparent isotropy on the macro-scale level.

The cell problem (21)–(23) is now formulated for the unit cube K_0 with the gas bubble of radius R_b in its center so that

$$V_0 = 1, \quad V_b = c = 4\pi R_b^3/3.$$

It is also relevant to introduce the local systems of the "fast" Cartesian and spherical coordinates X, Y, Z and R, θ, φ, respectively, with the origins at the cell center and with X-, Y-, Z-axes directed perpendicular to the cube sides.

The deviator $\dot{\mathbf{E}}$ is, then, fully presented by its diagonal components $\dot{E}_{xx}, \dot{E}_{yy}, \dot{E}_{zz}$ distinguished as q_x, q_y, q_z, respectively. By definition, the trace

$$q_x + q_y + q_z = 0,$$

and the rest of the components are equal to zero in accordance with the assumed orientation of the cell.

One could try different ways to construct approximations for $\dot{\mathbf{S}}$ and P on the basis of the simultaneous eqns (22), (23) in order to evaluate the integrals in eqns (24). But, to estimate the range of the uncertainty of resulting rheological relations, special attention should be paid to limiting cases in the directional deviation of the principal axes of the $\dot{\mathbf{S}}$-tensor approximation from those of the deviator $\dot{\mathbf{E}}$ (i.e., the Cartesian X-, Y-, Z-axes). Actually, this determines the magnitude of the term $\dot{\mathbf{E}}:\dot{\mathbf{S}}$ in the expression (21) for the second invariant of the tensor $\dot{\mathbf{e}}_0$ and, consequently, the value of η_0 in eqns (22), (24). With this in mind, let us consider two schemes of the deformation processes in the cell K_0 hereafter referred to as *spherical* and *cubical* approximations (see also Fig. 2).

4.1. *Spherical approximation*

If the volume concentration of bubbles c tends to zero ($c \to 0$), the radius of the bubble R_b in the reference cell K_0 (in the **R**-space) is the small parameter: $R_b = (3c/4\pi)^{1/3}$. Thus, the scale analysis of the cell problem (21)–(23) taking into account the new typical length can be performed. The direct integration of the continuity equation (22) over the fluid phase K_m and the second periodicity condition (23) yield the corresponding estimate for the velocity-fluctuation order

$$|\mathbf{W}| \sim 3|q_0|/(4\pi R_b^2) \sim |q_0|c^{-2/3}$$

around the bubble.

Hence, when $c \to 0$, the tensor $\dot{\mathbf{S}}$ of the volumetric expansion rates becomes the principal part of the tensor $\dot{\mathbf{e}}_0$ in eqn (21) in the vicinity of the bubble where $R = O(c^{1/3})$. Consequently,

$$II_{e0} \sim II_s \sim q_0^2/c^2, \quad \eta_0 \sim \eta(4q_0^2/c^2)$$

and the force balance eqns (22) give

$$P \sim p_b - p_m \sim \eta_0 q_0/c.$$

As a result, with the relative error of the order of $O(c)$ in the vicinity of the bubble the

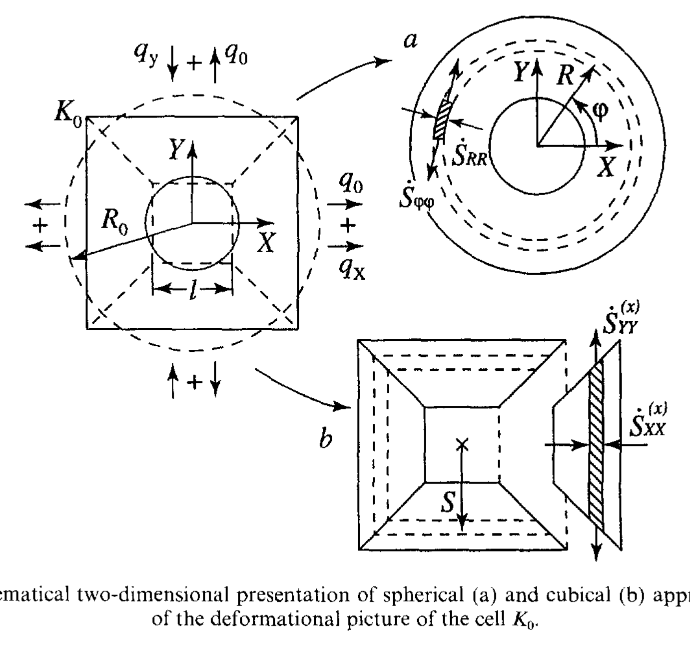

Fig. 2. Schematical two-dimensional presentation of spherical (a) and cubical (b) approximations of the deformational picture of the cell K_0.

solution P and $\mathbf{W}$ of the cell problem (22)–(23) does not depend on the tensor $\dot{\mathbf{E}}$ and the spherical symmetry prevails in the deformation picture, since it is close to the expansion (compression) of a single bubble in an infinite medium. This limiting case was the subject of the studies of Wilkinson and Ashby (1975), Salamatin *et al.* (1985) and also Salamatin and Lipenkov (1993). Thus, we write after them:

$$\mathbf{W} \equiv (q_0/c)(R_b/R)^3\mathbf{R} = q_0(R_0/R)^3\mathbf{R}, \tag{25}$$

where $R_0 = (3/4\pi)^{1/3}$ is the radius of the sphere of the unit volume equivalent to that of the cube K_0.

To motivate the use of the latter approximate presentation of $\mathbf{W}$ in a general situation, let us pay attention to the following detail. The principal directions of the corresponding strain rate tensor $\dot{\mathbf{S}}$ coincide with the orthonormal vector basis of the spherical coordinate system and seem to diverge maximally from the Cartesian axes directions. Hence, its extension to the whole cell K_0 as well as to arbitrary values of the bubble volume concentration c is supposed to lower the interaction between the deviatoric and volumetric strains on the macro-scale level, on the average.

Equation (25) yields the obvious relations for the spherical components of the $\dot{\mathbf{S}}$-tensor

$$\dot{S}_{RR} = -2q_0(R_0/R)^3, \quad \dot{S}_{\theta\theta} = \dot{S}_{\varphi\varphi} = q_0(R_0/R)^3,$$
$$\dot{S}_{R\theta} = \dot{S}_{R\varphi} = \dot{S}_{\theta\varphi} = \cdots = 0. \tag{26}$$

Substituting eqns (26) into eqns (21), one obtains:

$$II_E = (q_x^2+q_y^2+q_z^2)/2, \quad II_s = 3q_0^2(R_0/R)^6,$$
$$II_{e0} = II_E + II_S - 3q_0(R_0/R)^3\dot{E}_{RR}, \tag{27}$$

where the radial component $\dot{E}_{RR}$ of the deviator $\dot{\mathbf{E}}$ is given by the formula

$$\dot{E}_{RR} = (q_x\cos^2\varphi + q_y\sin^2\varphi)\sin^2\theta + q_z\cos^2\theta.$$

Further, let us replace the cube K_0 with the equivalent concentrical spherical cell of radius R_0 (see Fig. 2a). This permits representation of the averaging procedure $\langle\cdot\rangle_m$ by repeated integral with respect to the coordinate angles φ, θ and, then, radius R. So, to evaluate the integrals in eqns (24) it is sufficient to determine only the φ, θ-averaged excess pressure

$$\langle P\rangle_{\theta\varphi} = \frac{1}{4\pi}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}P\,\mathrm{d}\varphi. \tag{28}$$

To deduce the required relation for $\langle P\rangle_{\theta\varphi}$, we first write the momentum equation and the boundary condition on the bubble surface Γ_b (see eqn (22)) in the spherical coordinate system and, second, integrate in accordance with the averaging procedure (28) their projections onto the radial direction with respect to φ and θ. Finally, taking into account the facts that the tensor $\dot{\mathbf{e}}_0$ (i.e., $\dot{\mathbf{E}}$ and $\dot{\mathbf{S}}$) is a deviator and that all its components as functions of the coordinate angle φ are 2π-periodic, we straightforwardly come to the following boundary value problem:

$$\frac{\partial}{\partial R}\langle P\rangle_{\theta\varphi} = 2\frac{\partial}{\partial R}\langle(\dot{E}_{RR}+\dot{S}_{RR})\eta_0\rangle_{\theta\varphi} + \frac{6}{R}\langle(\dot{E}_{RR}+\dot{S}_{RR})\eta_0\rangle_{\theta\varphi},$$
$$[-\langle P\rangle_{\theta\varphi} + 2\langle(\dot{E}_{RR}+\dot{S}_{RR})\eta_0\rangle_{\theta\varphi}]_{|R=R_b} = p_m - p_b. \tag{29}$$

According to the structure of these equations, after substitution of eqns (9), (26), (27) it is relevant to designate $v = R/R_0$ and to write

$$\langle(\dot{E}_{RR}+\dot{S}_{RR})\eta_0\rangle_{\theta\varphi} = \langle(\dot{E}_{RR}-2q_0/v^3)\eta(12q_0^2/v^6+4II_E-12q_0\dot{E}_{RR}/v^3)\rangle_{\theta\varphi}$$
$$= -F(\sqrt{12}q_0/v^3\,;q_x,q_y,q_z)/\sqrt{3},$$

where the function $F(\xi)$ is defined as

$$F(\xi\,;q_x,q_y,q_z) = \langle(\xi-\sqrt{3}\dot{E}_{RR})\eta(\xi^2+4II_E-\sqrt{12}\xi\dot{E}_{RR})\rangle_{\theta\varphi}. \tag{30}$$

Then the use of eqn (30) and the immediate integration of the ordinary differential eqn (29) with account of the boundary condition at $R = R_b$ lead to the evident result:

$$\langle P\rangle_{\theta\varphi} = p_b-p_m-(2/\sqrt{3})F(\sqrt{12}q_0/v^3\,;q_x,q_y,q_z)$$
$$-\sqrt{12}\int_{c^{1/3}}^{v}F(\sqrt{12}q_0/\zeta^3\,;q_x,q_y,q_z)\frac{d\zeta}{\zeta},\quad v = R/R_0. \tag{31}$$

4.2. *Cubical approximation*

Now let us note that the cube K_0 is composed of six identical, regular, pyramidal sectors, whose bases are the cube sides, and whose summits lie in the cell center (see Fig. 2). In each sector the principal axes of the expansion strain rate tensor $\dot{\mathbf{S}}$ are close by directions to the Cartesian axes: the radial axis being roughly colinear with the Cartesian unit vector perpendicular to the base of the sector. Thus, another limiting scheme of the deformations shown in Fig. 2b arises.

First, we change the spherical bubble to the equivalent cubic one with sides perpendicular to the X-, Y-, Z-axes and with the edge length $l = c^{1/3}$. Second, we assume that in each sector in cross sections parallel to its base the strain rate tensor $\dot{\mathbf{S}}$ represents a simple three-dimensional deformation. So, for a certain sector α with the base parallel to $\beta\gamma$-coordinate plane ($\alpha,\beta,\gamma = X, Y, Z$ in circular permutation, $\alpha \neq \beta \neq \gamma$) approximately only the diagonal components of the $\dot{\mathbf{S}}$-tensor (distinguished further as $\dot{\mathbf{S}}^{(\alpha)}$) are non-zero and are written as follows:

$$\dot{S}^{(\alpha)}_{\alpha\alpha} = -2q_0/(2S)^3,\quad \dot{S}^{(\alpha)}_{\beta\beta} = \dot{S}^{(\alpha)}_{\gamma\gamma} = q_0/(2S)^3, \tag{32}$$

where S is the distance from the cell center along the α-sector axis (α-coordinate axis). Let us also designate the excess pressure in the α-section as $P^{(\alpha)}$.

In the framework of this approximation, in all sectors, deformation characteristics are functions of S only. It is clear that, in general case, $\dot{\mathbf{S}}^{(\alpha)}$ and $P^{(\alpha)}$ are different for different α and this makes $\dot{\mathbf{S}}$ and P discontinuous at the side faces of the neighboring sectors. Consequently, in accordance with eqns (21), (32) for each α-sector the cell problem (22) is reduced to the momentum balance equation (its projection onto the α-coordinate axis) written for an arbitrary cross section and to the boundary condition at the cubic-bubble side.

$$\frac{\partial P^{(\alpha)}}{\partial S} = 2\frac{\partial}{\partial S}[(q_\alpha-2q_0/(2S)^3)\eta_\alpha]+[4(q_\alpha-2q_0/(2S)^3)\eta_\alpha$$
$$-2(q_\alpha+q_0/(2S)^3)(\eta_\beta+\eta_\gamma)-2P^{(\alpha)}+P^{(\beta)}+P^{(\gamma)}]/S,$$
$$P^{(\alpha)}_{|s=l/2} = p_b-p_m+2(q_\alpha-2q_0/c)\eta_{\alpha|s=l/2}. \tag{33}$$

Here

$$\eta_\alpha = \eta(4II_{c0}^{(\alpha)}), \quad II_{c0}^{(\alpha)} = II_E + II_S - 3q_\alpha q_0/(2S)^3, \quad II_S = 3q_0^2/(2S)^6. \tag{34}$$

The terms in square brackets on the right-hand side of the momentum eqn (33) are responsible for the divergence of the sector domain in α-axis direction and the force interaction with the neighboring sectors.

Again there is no necessity to solve all the simultaneous eqns (33) for different α to evaluate the integrals in eqns (24). Following the same line of considerations, as in the previous Section 4.1, we define the averaged sector pressure by the procedure

$$\langle P^{(\beta)}\rangle_\beta = (P^{(X)} + P^{(Y)} + P^{(Z)})/3.$$

Then, adding eqns (33) and introducing the function

$$F(\xi; q_x, q_y, q_z) = \langle(\xi - \sqrt{3}q_\beta)\eta(\xi^2 + 4II_E - \sqrt{12}\xi q_\beta)\rangle_\beta \tag{35}$$

analogous with that in eqn (30), we come to the similar problem, as for eqns (29). The final relation for $\langle P^{(\beta)}\rangle_\beta$ is identical to eqn (31) with $v = 2S$.

Approximations (32)–(35), together with eqn (31), give the upper estimate of the interaction between volumetric expansion and deviatoric strains in bubbly medium on the micro-scale level.

5. MACRO-SCALE RHEOLOGICAL RELATIONS FOR ISOTROPIC BUBBLY MEDIUM

The final stage of the theoretical study is to evaluate and to analyze eqns (24) on the basis of the schematic presentations (26), (32) for $\dot{\mathbf{E}}$ and eqns (27), (34) for II_{c0}, using also eqns (30) and (35) in the deduced relation (31) for $\langle P\rangle_{\theta\varphi}$ and $\langle P^{(\beta)}\rangle_\beta$.

Let us start with the first eqn (24). The integral procedure $\langle\cdot\rangle$ for the spherical or cubical approximations can be written in the unified form:

$$\langle\cdot\rangle_m = 3\int_{c^{1/3}}^{1} v^2\langle\cdot\rangle\,\mathrm{d}v, \tag{36}$$

where $\langle\cdot\rangle$ denotes either directional averaging $\langle\cdot\rangle_{\theta\varphi}$ with $v = R/R_0$ or sector averaging $\langle\cdot\rangle_\beta$ with $v = 2S$, respectively.

Thus, substituting eqn (31) into eqn (36) for $\langle P\rangle_m$ and integrating its last integral term by parts, we eventually obtain from the first eqn (24) the general relationship between the gas-medium pressure drop and the expansion rate q_0.

$$p_b - p_m = \frac{2}{\sqrt{3}}\int_{\sqrt{12}q_0}^{\sqrt{12}q_0/c} F(\xi; q_x, q_y, q_z)\frac{\mathrm{d}\xi}{\xi}, \tag{37}$$

which is additionally transformed by the change of variable: $\xi = \sqrt{12}q_0/\zeta^3$.

In accordance with eqns (30) and (35) it is also influenced by the deviatoric strain rates q_x, q_y, q_z.

Now the same change of variable in the integral in eqn (31) and the substitution of eqn (37) give the final formula for the directionally averaged pressure fluctuations within the cell

$$\langle P\rangle = \frac{2}{\sqrt{3}}\left[\int_{\sqrt{12}q_0}^{\sqrt{12}q_0/v^3} F(\xi; q_x, q_y, q_z)\frac{\mathrm{d}\xi}{\xi} - F\left(\frac{\sqrt{12}q_0}{v^3}; q_x, q_y, q_z\right)\right], \quad c^{1/3} \leqslant v = R/R_0, 2S \leqslant 1. \tag{38}$$

Passing to the second eqn (24), it is important to underline the fact that the principal axes of the macro-stress tensor $\mathbf{T}$ coincide with those of the macro-strain rate deviator $\dot{\mathbf{E}}$ (i.e., with the X-, Y-, Z-coordinate axes). This is true by definition for the cubic approximation (see eqns (32)). As for the spherical scheme of deformations in the cell K_0 with the $\dot{\mathbf{S}}$-tensor given by eqns (26), the following presentation is valid in the Cartesian coordinate system:

$$S_{xx} = \frac{q_0}{v^3}(1-3\sin^2\theta\cos^2\varphi), \quad S_{yy} = \frac{q_0}{v^3}(1-3\sin^2\theta\sin^2\varphi),$$

$$S_{zz} = \frac{q_0}{v^3}(1-3\cos^2\theta), \quad S_{xy} = -\frac{q_0}{v^3}\sin^2\theta\sin^2\varphi, \ldots, \quad v = R/R_0. \tag{39}$$

So, the straightforward verification shows that, for example, the mean value $\langle\eta_0 S_{xy}\rangle_{\theta\varphi}$ evaluated in accordance with the averaging procedure (28) for η_0 determined by eqns (9), (27) is equal to zero. Consequently, eqn (36) gives $\langle\eta_0 S_{xy}\rangle_m = 0$, and the second eqn (24) results in $T_{xy} = 0$. The same holds for all $T_{\alpha\beta}$, when $\alpha \neq \beta$. Hence, only the diagonal components T_{xx}, T_{yy}, T_{zz} are to be specified.

In the general case, the decomposition (36) and the substitution of eqns (27), (39) or eqns (32), (34) into the second eqn (24) yield the required rheological relations:

$$T_{\alpha\alpha} = 2q_\alpha \int_c^1 H(\sqrt{12}q_0/\zeta\,; q_x, q_y, q_z)\,\mathrm{d}\zeta$$

$$+2q_0 \int_c^1 G_\alpha(\sqrt{12}q_0/\zeta\,; q_x, q_y, q_z)\frac{\mathrm{d}\zeta}{\zeta}, \quad \alpha = x, y, z. \tag{40}$$

It should be mentioned that the integrals in eqn (40) have been transformed by the change of variables: $\zeta = v^3$. New functions H and G_α are the direct analogs of the function F introduced by eqn (30) or by eqn (35). In fact, we have

(1) for spherical approximation

$$\begin{aligned}
H(\xi\,; q_x, q_y, q_z) &= \langle\eta(\xi^2+4II_E-\sqrt{12}\xi\dot{E}_{RR})\rangle_{\theta\varphi},\\
G_x(\xi\,; q_x, q_y, q_z) &= \langle(1-3\sin^2\theta\cos^2\varphi)\eta(\xi^2+4II_E-\sqrt{12}\xi\dot{E}_{RR})\rangle_{\theta\varphi},\\
G_y(\xi\,; q_x, q_y, q_z) &= \langle(1-3\sin^2\theta\sin^2\varphi)\eta(\xi^2+4II_E-\sqrt{12}\xi\dot{E}_{RR})\rangle_{\theta\varphi},\\
G_z(\xi\,; q_x, q_y, q_z) &= \langle(1-3\cos^2\theta)\eta(\xi^2+4II_E-\sqrt{12}\xi\dot{E}_{RR})\rangle_{\theta\varphi}\,;
\end{aligned} \tag{41}$$

(2) for cubical approximation

$$\begin{aligned}
H(\xi\,; q_x, q_y, q_z) &= \langle\eta(\xi^2+4II_E-\sqrt{12}\xi q_\beta)\rangle_\beta,\\
G_\alpha(\xi\,; q_x, q_y, q_z) &= \langle(1-3\delta_{\alpha\beta})\eta(\xi^2+4II_E-\sqrt{12}\xi q_\beta)\rangle_\beta, \quad \alpha = x, y, z,
\end{aligned} \tag{42}$$

where $\delta_{\alpha\beta}$ are the components of the identity tensor $\boldsymbol{\delta}$.

The right-hand side of eqn (40) evidently depends on q_0, and the deviatoric macro-stresses in bubbly medium are controlled by density relaxation processes.

Equations (37), (38), (40) can be considerably simplified, if the sum ξ^2+4II_E is a dominant part in the argument of the function η in eqns (30), (41) and (35), (42). When $\xi^2+4II_E \gg \sqrt{12}\xi\dot{E}_{RR}$, the following transformations based on Taylor formula are valid for eqn (30):

$$F(\xi, q_x, q_y, q_z) \approx \langle (\xi - \sqrt{3}\dot{E}_{RR})[\eta(\xi^2+4II_E) - \sqrt{12}\xi\dot{E}_{RR}\eta'(\xi^2+4II_E)]\rangle_{\theta\varphi}$$
$$= [\eta(\xi^2+4II_E) + 6\langle \dot{E}^2_{RR}\rangle_{\theta\varphi}\eta'(\xi^2+4II_E)]\xi$$
$$= [\eta(\xi^2+4II_E) + 4\kappa II_E\eta'(\xi^2+4II_E)]\xi, \quad \kappa = 0.4.$$

To obtain this relation it has also been taken into account that $\langle \dot{E}_{RR}\rangle_{\theta\varphi} = 0$ and $\langle \dot{E}^2_{RR}\rangle_{\theta\varphi} = (4/15)II_E$.

Equation (35) can easily be converted into the same form but the value of the factor κ is different: $\kappa = 1$. Thus, instead of eqn (37) we write, approximately:

$$p_b - p_m = \frac{2}{\sqrt{3}}\int_{\sqrt{12}q_0}^{\sqrt{12}q_0/c} [\eta(\xi^2+4II_E) + 4\kappa II_E\eta'(\xi^2+4II_E)]\,\mathrm{d}\xi, \quad 0.4 \leqslant \kappa \leqslant 1. \tag{43}$$

Here the upper and lower bounds for κ correspond to the two limiting schemes of the cell deformations considered above.

Now let us examine eqns (40)–(42). It is obvious that, when ξ^2+4II_E is a larger part of the η-function argument in eqns (41), (42), in both cases we have

$$H(\xi, q_x, q_y, q_z) \approx \eta(\xi^2+4II_E).$$

As for the functions G_α, the straightforward calculations lead to the following simplified presentation

$$G_\alpha(\xi, q_x, q_y, q_z) \approx \sqrt{12}\kappa\xi\eta'(\xi^2+4II_E)q_\alpha, \quad \alpha = x, y, z,$$

where $\kappa = 0.4$ or 1 for spherical or cubical approximations, respectively.

Finally, eqns (40) take the form:

$$\mathbf{T} = 2\left\{\int_c^1 \left[\eta\left(\frac{12q_0^2}{\zeta^2}+4II_E\right) + \kappa\frac{12q_0^2}{\zeta^2}\eta'\left(\frac{12q_0^2}{\zeta^2}+4II_E\right)\right]\mathrm{d}\zeta\right\}\dot{\mathbf{E}}, \quad 0.4 \leqslant \kappa \leqslant 1. \tag{44}$$

It should be emphasized that, in case of a linear function $\eta(\xi)$, eqns (43), (44) are the exact replications of eqns (37), (40). Therefore they are thought to be in good agreement with eqns (37), (40) in general, unless the nonlinearity of the constitutive relation (1) on the micro-scale level is too strong. Possible variations of the factor κ within the limits from 0.4 to 1 still cover the range of their uncertainty. Moreover, κ can be used for adjusting the theoretical formulas (43), (44) to experimental data and for tuning them to simulate rheological behavior of real bubbly mediums.

6. BUBBLY ICE RHEOLOGY. DISCUSSION

For the purpose of applications and discussion we aim at isotropic polycrystalline ice as a widely spread substance, which approximately follows the creep flow law (1) and has rheological properties similar to those of metals at high temperatures (Glen, 1955).

A representative collection of various experiments considered by Budd (1969) reveals a power (polynomial) form of the relationship between the apparent strain rate $\dot{e}_a = \sqrt{II_e}$ and stress $\tau_a = \sqrt{II_\tau}$ in pure ice:

$$2\dot{e}_a = \left(\frac{\tau_a}{\mu_1} + \frac{\tau_a^\alpha}{\mu_2}\right)\exp\left[\frac{Q}{R_sT_s}\left(\frac{T-T_s}{T}\right)\right]. \tag{45}$$

Here α is the creep index, μ_1 and μ_2—the rheological coefficients (constants), Q—the apparent activation energy, R_s—the gas constant ($R_s = 8.314$ J/(mole·K)), T_s—the standard temperature ($T_s = 273.15$ K).

Further, for computations we set $\alpha = 3$, $\mu_1 = 0.8$ MPa·yr, $\mu_2 = 0.004$ MPa$^\alpha$·yr, $Q = 60$ kJ/mole that is close to estimates deduced by Budd (1969) and Sumskii (1969) from the experimental data.

The temperature of ice hereafter is fixed to be $T = 223.15$ K, i.e., -50°C. The latter assumption does not limit possibilities of discussion, since the exponential factor in eqn (45) merely scales the strain rates. As for the range of stresses, it is chosen to cover the extreme cases of the linear and power asymptotics of the law (45) for small and large τ_a, respectively, as well as intermediate values, for which both terms in eqn (45) are significant.

Air bubble volume concentration in natural glacier ice, as a rule, does not exceed 0.1 (Salamatin *et al.*, 1985).

To use the rheological law (45) in eqns (37), (38), (40) and in their simplified analogs (43), (44) we need the function $\eta(\xi)$. With this aim the flow law (1) can be easily rewritten in terms of apparent stress τ_a and strain rate $\dot{e}_a$: $\tau_a = 2\eta(4\dot{e}_a^2)\dot{e}_a$. So, designating the argument of the rheological coefficient η as ξ (i.e., taking $2\dot{e}_a = \sqrt{\xi}$) and substituting the resulting expression for τ_a into eqn (45), we come to the following equation, which identifies $\eta(\xi)$:

$$(1/M_1 + (\xi^{1/2}\eta)^{\alpha-1}/M_2)\eta = 1,$$

where M_1 and M_2 are the linear and non-linear viscosities μ_1 and μ_2, respectively, divided by the exponential factor.

On the micro-scale level the creep flow model (1) and its particular case given by eqn (45) do not depend on the third deviatoric strain rate invariant III_e. This is not true, in general, for the macro-scale processes governed by eqns (37), (38), (40) although holds approximately for eqns (43), (44). So, to compare different patterns of deviatoric deformations with the same apparent macro-strain rate $\dot{E}_a = \sqrt{II_E}$, but different values of the third invariant III_E, we have considered:

(a) simple three-dimensional extension, $q_x = q_y = \dot{E}_a/\sqrt{3}, q_z = -2\dot{E}_a/\sqrt{3}$ ($III_E = -2\dot{E}_a^3/\sqrt{27}$);

(b) simple three-dimensional compression, $q_x = q_y = -\dot{E}_a/\sqrt{3}, q_z = 2\dot{E}_a/\sqrt{3}$ ($III_E = 2\dot{E}_a^3/\sqrt{27}$);

(c) simple two-dimensional extension, $q_x = -q_y = \dot{E}_a, q_z = 0$ ($III_E = 0$).

Typical variations of the averaged excess pressure $\langle P \rangle$ within a bubble cell in accordance with eqn (38) at $c = 0.02$ in different situations are depicted in Fig. 3. For comparatively small deviatoric deformations ($\dot{E}_a \ll q_0/c$) the simplified expressions for F are accurate to 1–3%, $\langle P \rangle$ does not depend on the way of deformation and the graphs for cubical and spherical approximations are not distinguishable (see curve 1). If the deviatoric strain rates increase and reach the same order as the volumetric expansion rate in the bubble vicinity ($\dot{E}_a \sim q_0/c$), the cubical approximation becomes sensitive to the patterns of deformations (curves 2a, b for simple three-dimensional extension and compression), while the spherical scheme remains extremely conservative (curve 3). In case (c) of simple two-dimensional extension the corresponding curve is very close to the arithmetic mean of the cases (a) and (b) which are not shown in the figure. This comparison reveals strong interaction between bubble pressure (density) relaxation and macro-scale flow in bubbly non-Newtonian mediums. It also points out the possible development of stagnation zones around bubbles. At the same time it is clear that the spherical and cubical approximations are just exaggerated schematic models of the real deformation picture within cells and should be regarded as limiting estimations. The dashed lines in Fig. 3 correspond to the simplified relations for the function F and represent pressure distribution in the cell on the average.

The gas-medium pressure drop $p_b - p_m$ given by eqn (37) is plotted in Fig. 4 against macro-scale expansion rate q_0. Curves 1–3 illustrate the influence of the deviatoric strain rate $\dot{E}_a$ on the relationship at $c = 0.02$ for cubic approximation. As it may have been

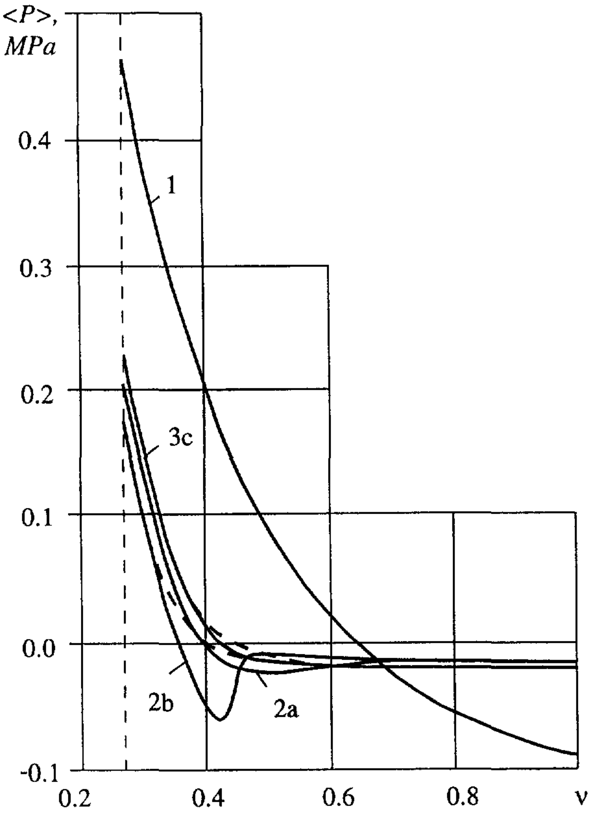

Fig. 3. Directionally averaged excess pressure $\langle P \rangle$ as a function of normalized distance ν from the cell center within ice matrix at $c = 0.02$ and $q = 2 \cdot 10^{-4}$ yr^{-1}: (1) $E_a = 2 \cdot 10^{-4}$ yr^{-1} (the same curve for cubical and spherical approximations for all patterns of deformation); (2) and (3) $E_a = 4 \cdot 10^{-3}$ yr^{-1} for cubical and spherical approximations, respectively. Cases (a), (b), and (c) are: simple 3-d extension, 3-d compression, and 2-d extension, respectively. The dashed lines correspond to the simplified forms (30) and (35) of the function F at $\kappa = 0.4$ and 1.

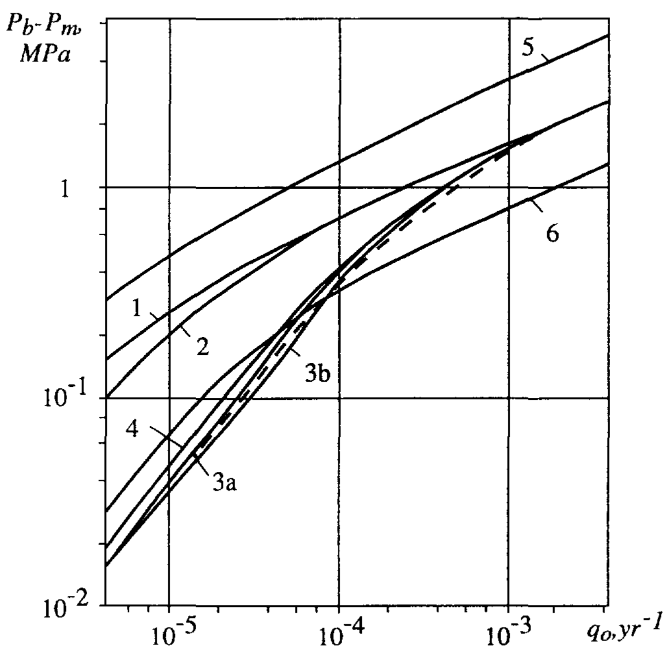

Fig. 4. Phase pressure drop p_b–p_m vs volumetric expansion rate q_0 in bubbly ice: (1)–(3) for cubical approximation at $c = 0.02$ with $E_a = 10^{-5}$, $2 \cdot 10^{-4}$, $4 \cdot 10^{-3}$ yr^{-1}, respectively; (4) for spherical approximation at $c = 0.02$ with $E_a = 4 \cdot 10^{-3}$ yr^{-1}; (5), (6) for all patterns of deformations and schematic approximations with $E_a = 2 \cdot 10^{-4}$ yr^{-1} at $c = 0.005$, 0.08, respectively. Cases (a) and (b) are simple 3-d extension and 3-d compression. The dashed line corresponds to the simplified relation (43) at $\kappa = 1$.

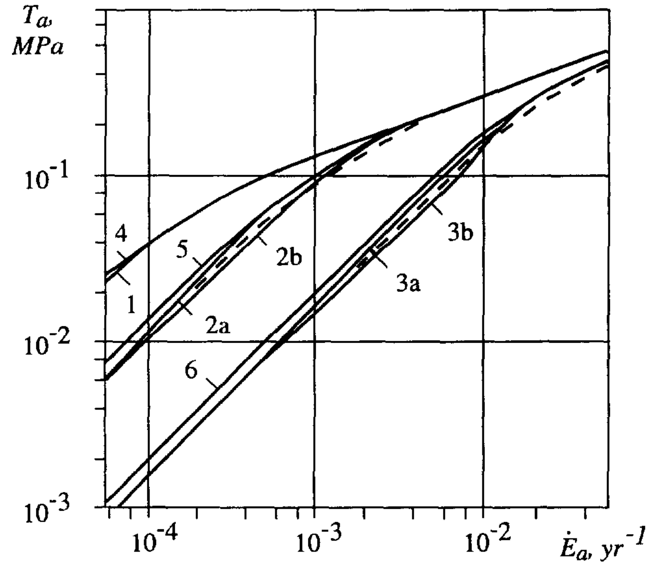

Fig. 5. Creep law of bubbly ice at $c = 0.02$: (1)–(3) for cubical and (4)–(6) for spherical approximations at $q_0 = 10^{-5}, 2 \cdot 10^{-4}, 4 \cdot 10^{-3}$ yr^{-1}, respectively. Cases (a) and (b) are simple 3-d extension and 3-d compression. The dashed lines correspond to the simplified relation (44) at $\kappa = 1$.

expected, only at large $\dot{E}_a$ ($\sim q_0/c$) simple three-dimensional extension and compression result in slightly different curves 3a and 3b, respectively. The dashed line is the relation (43) at $\kappa = 1$. In the latter case the spherical approximation leads to curve 4 practically the same for all patterns of deformation and for both eqns (37) and (43). The discrepancy between the two deformation schemes in the resulting rheological law relating pressure drop $p_b - p_m$ to volumetric expansion rate q_0 is not great (compare curves 3, 4, and the dashed one in Fig. 4) and diminishes as $\dot{E}_a$ decreases. Thus, eqn (43) provides a reliable basis for modelling pressure (density) relaxation in bubbly mediums on macro-scale level. In particular, curves 5 and 6 in Fig. 4 show strong dependence of density relaxation rates on bubble volume concentration c.

The graphs plotted in Fig. 5 present the macro-scale flow law (40) of bubbly ice: the dependence of the apparent stress $T_a = \sqrt{II_T}$ on the apparent deviatoric strain rate $\dot{E}_a = \sqrt{II_E}$. For small volume concentration of bubbles, the main factor that controls the rheological behavior of ice is the macro-scale expansion rate q_0 (see curves 1–3 for cubical approximation). Variations of c within the range $0.005 < c < 0.08$ do not change, for instance, curves 3 at $c = 0.02$ more than by 2%. Again, different patterns of deformation influence noticeably (but not much) the creep flow law only if the deformation rates $\dot{E}_a$ and q_0/c in bubble vicinities are comparable: curves 2a, b and 3a, b. The case (c) of simple two-dimensional extension is the arithmetic mean of cases (a) and (b). The dashed lines correspond to eqn (44) that is evidently valid for cubical approximation ($\kappa = 1$) on the average. As for the spherical approximation, at each value of q_0 we obtain practically single curves 4–6, which coincide with accuracy to 2% with eqn (44) at $\kappa = 0.4$. Thus, eqn (44), as well as eqn (43) determining the phase pressure drop, appears to be a plausible form of the macro-scale flow law of bubbly ice. An appropriate value of the tuning parameter κ should finally be found from experimental data.

7. CONCLUSION

A quasi-periodic model of a two-phase, bubbly medium is a useful physical basis to simulate rheological behaviour of the mixture on macro-scale level, regarding it as continuous substance with apparent rheophysical properties. Such an approach permits us to employ the vehicle of perturbation methods and asymptotic analysis of periodic structures to derive rigorously averaged balance equations governing flow and density (pressure) relaxation of bubbly medium in macrocontinual approximation. The "dual" cell problem leads to corresponding rheological laws. The latter relations have an integral form and even a plausible schematic solution of the cell problem results in reliable constitutive equations and, totally, in the complete macro-scale model.

This general procedure has been applied to describe creep flow and relaxation compressibility of non-Newtonian fluid with a large number of small gas inclusions. Strong interaction between deviatoric strains and volumetric expansion (compression) is revealed. Rheological relations determining the constitutive part of the macro-stress tensor and the phase pressure drop in bubbly medium are written in explicit form and contain one parameter, which variation range represents the uncertainty of the approximate solutions of the cell problem. At the same time this parameter may be used to adjust the theoretical rheological equations to original experimental data in order to compensate inevitable model errors.

The discussion of these results in application to the bubbly ice rheology shows self-consistency of the limiting approximate schemes of cell deformations used for the construction of the rheological model. Within the limits of the existing uncertainties bubbly ice behavior on the macro-scale level remains independent of the third invariant of the strain rate deviator.

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